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Maximum principle for differential games of forward–backward stochastic systems with applications ☆

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ABSTRACT

This paper is concerned with a maximum principle for both zero-sum and nonzero-sum games. The most distinguishing feature, compared with the existing literature, is that the game systems are described by forward–backward stochastic differential equations. This kind of games is motivated by linear-quadratic differential game problems with generalized expectation. We give a necessary condition and a sufficient condition in the form of maximum principle for the foregoing games. Finally, an example of a nonzero-sum game is worked out to illustrate that the theories may find interesting applications in practice. In terms of the maximum principle, the explicit form of an equilibrium point is obtained.

1. Introduction

This work is interested in the stochastic differential games of systems of forward–backward stochastic differential equations (FBSDEs, for short). One of the motivations of this study is the problem of finding a saddle point in a linear quadratic (LQ, for short) zero-sum differential game, where the performance criterion is defined by generalized expectation. We now explain this in more detail.

We consider the following controlled linear stochastic differential equation (SDE, for short):

$$\begin{cases} dX(t) = \big[A_1X(t) + B_1v_1(t) + C_1v_2(t)\big]dt + \big[A_2X(t) + B_2v_1(t) + C_2v_2(t)\big]dB(t), \\ X(0) = X_0, \end{cases} \tag{1}$$

with the performance criterion

$$J\big(v_1(\cdot), v_2(\cdot)\big) = \frac{1}{2}\mathbb{E}\left\{\big\langle GX(T), X(T)\big\rangle + \int_0^T \big[\big\langle A_3X(t), X(t)\big\rangle + \big\langle B_3v_1(t), v_1(t)\big\rangle + \big\langle C_3v_2(t), v_2(t)\big\rangle\big]\,dt\right\}. \tag{2}$$

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We assume that (1) and (2) are well posed. In general, the classical stochastic LQ zero-sum differential game problem is formulated as follows:

Problem $(LQ)^2_0$. For any $X_0 \in \mathrm{R}^n$, find a $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2[0,T]$ such that

$$\begin{aligned} J\big(u_1(\cdot), u_2(\cdot)\big) &= \sup_{v_2(\cdot)\in\mathcal{U}_2[0,T]} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1[0,T]} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) \\ &= \inf_{v_1(\cdot)\in\mathcal{U}_1[0,T]} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2[0,T]} J\big(v_1(\cdot), v_2(\cdot)\big)\Big), \end{aligned}$$

where $\mathcal{U}_1 \times \mathcal{U}_2[0,T]$ denotes certain admissible control set.

Remark 1.1. The expectation $\mathbb{E}$ in (2) is a linear expectation and does not express people's *preferences* or criterion involving *risk* (see [1,4,5] and the references therein). One alternative way is to use the so-called generalized expectation induced by backward stochastic differential equation (BSDE, for short) to characterize the performance criterion (see Peng [15,16]), because generalized expectation satisfies all the properties that expectation $\mathbb{E}$ has, except for its linearity.

Consider the following BSDE:

$$\begin{cases} d\eta(t) = -g\big(\zeta(t)\big)\,dt + \zeta(t)\,dB(t), \\ \eta(T) = \xi. \end{cases} \tag{3}$$

Under certain conditions, Eq. (3) has a unique solution $(\eta(\cdot), \zeta(\cdot))$. In addition to condition $g(\zeta) = 0 \Leftrightarrow \zeta = 0$, we define

$$\mathcal{E}_g(\xi) = \eta(0). \tag{4}$$

$\mathcal{E}_g(\xi)$ is called the generalized expectation (also called g-expectation) of ξ associated with g (see Peng [15]). With the above generalized expectation, we introduce the following new performance criterion

$$J_g\big(u_1(\cdot), u_2(\cdot)\big) = \frac{1}{2}\mathcal{E}_g\left(\big\langle GX(T), X(T)\big\rangle + \int_0^T \big[\big\langle A_3X(t), X(t)\big\rangle + \big\langle B_3v_1(t), v_1(t)\big\rangle + \big\langle C_3v_2(t), v_2(t)\big\rangle\big]\,dt\right), \tag{5}$$

and formulate stochastic LQ differential game problem with generalized expectation as follows:

Problem $(LQ)^2_g$. For any $X_0 \in \mathrm{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}^2[0,T]$ such that

$$\begin{aligned} J_g\big(u_1(\cdot), u_2(\cdot)\big) &= \sup_{v_2(\cdot)\in\mathcal{U}_2[0,T]} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1[0,T]} J_g\big(v_1(\cdot), v_2(\cdot)\big)\Big) \\ &= \inf_{v_1(\cdot)\in\mathcal{U}_1[0,T]} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2[0,T]} J_g\big(v_1(\cdot), v_2(\cdot)\big)\Big). \end{aligned} \tag{6}$$

In order to connect *Problem* $(LQ)^2_g$ with the game problem of forward–backward stochastic systems, we construct a decoupled FBSDE:

$$\begin{cases} dX(t) = \big[A_1X(t) + B_1v_1(t) + C_1v_2(t)\big]\,dt + \big[A_2X(t) + B_2v_1(t) + C_2v_2(t)\big]\,dB(t), \\ dY(t) = -\big[g\big(Z(t)\big) + \big\langle A_3X(t), X(t)\big\rangle + \big\langle B_3v_1(t), v_1(t)\big\rangle + \big\langle C_3v_2(t), v_2(t)\big\rangle\big]\,dt + Z(t)\,dB(t), \\ X(0) = X_0, \quad Y(T) = \dfrac{1}{2}\big\langle GX(T), X(T)\big\rangle, \end{cases} \tag{7}$$

with the new performance criterion

$$\begin{aligned} \bar{J}\big(v_1(\cdot), v_2(\cdot)\big) = \mathbb{E}\Bigg\{ & \int_0^T \big[g\big(Z(t)\big) + \big\langle A_3X(t), X(t)\big\rangle + \big\langle B_3v_1(t), v_1(t)\big\rangle \\ & + \big\langle C_3v_2(t), v_2(t)\big\rangle\big]\,dt + \frac{1}{2}\big\langle GX(T), X(T)\big\rangle \Bigg\}. \end{aligned} \tag{8}$$

Then *Problem* $(LQ)^2_g$ is equivalent to max-minimizing the performance criterion (8) over $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2[0,T]$, subject to FBSDE (7). Namely, it can be transformed into a differential game problem with the system of FBSDE (7) and performance criterion (8). Then differential game problems of forward–backward stochastic systems come out quite naturally and deserve further research.

Since the foundations of game theory were presented by Von Neumann and Morgenstern [21], the importance of strategic behavior in theory and practice has been increasingly recognized and many authors have joined in the study of differential games. Game theory pervading economic theory, attracts more and more research attentions, and is used widely in other social and behavioral sciences. It is also a very good tool which helps us understand economic, social, political, and biological phenomena. Game theoretic methods gradually dominate microeconomic theory and are used in many other fields of economics and a wide range of other social and behavioral sciences. For more information, refer to J. Osborne [17]. There is a large amount of literature about game theory, for example, Kieu An and Øksendal [2], Hamadène [6], Jiang [7,8], Jimenez-Lizarraga and Fridman [9], Jimenez-Lizarraga and Basin [10], Konishi et al. [11], Lim and Zhou [12], Wu [23], Yong [26], Yeung and Petrosjan [25], etc. However, the game systems they studied are either ODEs or SDEs, not involve more general FBSDEs.

The theory of FBSDEs has received considerable research attention in recent years because of its interesting structure and its usefulness in various applied areas such as dynamic risk measure, stochastic optimal control, stochastic differential utility, option pricing, contingent claim valuation, and second order partial differential equation theories. For more information, refer to Ma and Yong [13], Yong and Zhou [28], and the references therein. Due to the well-defined dynamics of FBSDEs and the broad applications of game theory, it is very natural and appealing, at the theoretical level, to consider the game problems of FBSDEs.

To our best knowledge, there are not many studies on optimal control of FBSDEs. There are only a few papers dealing with this class of control problems (see, e.g., Peng [14], Shi and Wu [19], Xu [24] and Yong [27] for non-random jumps, and Øksendal and Sulem [18] and Shi and Wu [20] for random jumps), but there are few papers on game theories of FBSDEs and performance criteria. Up till now, there are only two papers about differential games of BSDEs: one is Yu and Ji [30], where an LQ nonzero-sum game was studied by a standard completion of squares techniques and the explicit form of a Nash equilibrium point was obtained; the other one is Wang and Yu [22], where the game system was a nonlinear BSDE, and a necessary condition and a sufficient condition in the form of maximum principle were established, respectively.

However, the game problems mentioned above are restricted to forward (stochastic) or backward stochastic systems. To our best knowledge, there are only two papers about the differential games of forward–backward stochastic systems (see Buckdahn and Li [3] and Yu [29]). In Buckdahn and Li [3], the game system was described by a decoupled FBSDE, and the performance criterion was defined by the solution variable of BSDE, at the value at time 0. Buckdahn and Li proved a dynamic programming principle for both the upper and the lower value functions of the game, and showed that these two functions are the unique viscosity solutions to the upper and the lower Hamilton–Jacobi–Bellman–Isaacs equations, but they neither considered the nonzero-sum games problem nor initiated a study of a necessary condition and a sufficient condition in the form of maximum principle. Recently, Yu [29] studied a linear-quadratic case of nonzero-sum game problem for forward–backward stochastic systems, where the FBSDE method was employed to obtain an explicit Nash equilibrium point. However, in the present paper, we shall study the problem in the general situation.

This paper is organized as follows. In Section 2, we formulate the zero-sum and nonzero-sum games of forward–backward stochastic systems, respectively. In Section 3, combining FBSDE theory with certain classical convex variational techniques, we prove a necessary condition and a sufficient condition for the foregoing game problems in the form of maximum principle. In Section 4, an example of a nonzero-sum differential game is worked out to illustrate theoretical applications. In terms of maximum principle, the explicit form of an equilibrium point is obtained. Finally, this paper provides some concluding remarks.

2. Formulation of the problem

Let T be a fixed constant and $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T}, P)$ be a complete filtered probability space, on which a standard Brownian motion $B(\cdot) \in \mathrm{R}^d$ is defined with $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ being the P-completed natural filtration generated by $B(\cdot)$ and $\mathscr{F} \triangleq \mathscr{F}_T$. Let $|x|$ denote the Euclidean norm of $x \in \mathrm{R}^n$ and $\langle x, y\rangle$ be the inner product of $x, y \in \mathrm{R}^n$. The transpose and Euclidean norm of a matrix $M = (m^{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d} = (m^1, \ldots, m^d) \in \mathrm{R}^{n\times d}$ are expressed as M^* and $|M| = \sqrt{\mathrm{trace}(MM^*)}$, respectively. Similarly, $\langle M_1, M_2\rangle = \mathrm{trace}(M_1 M_2^*)$ with $M_1, M_2 \in \mathrm{R}^{n\times d}$. We also introduce the following three spaces of processes which will be frequently used in the sequel:

$$\mathcal{L}^p_{\mathscr{F}_T}\big(\Omega; \mathrm{R}^k\big) \triangleq \big\{\xi : \Omega \to \mathrm{R}^k \bigm| \xi \text{ is } \mathscr{F}_T\text{-measurable}, \ \mathbb{E}|\xi|^p < +\infty\big\};$$

$$\mathcal{S}^p\big(0, T; \mathrm{R}^n\big) \triangleq \Big\{\phi(\cdot) \Bigm| \phi(\cdot) \text{ is an } \mathrm{R}^n\text{-valued } \mathscr{F}_t\text{-adapted cà dlàg process: } \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \big|\phi(t)\big|^p\Big] < +\infty\Big\};$$

$$\mathcal{H}^p\big(0, T; \mathrm{R}^{n\times d}\big) \triangleq \left\{\phi(\cdot) \Bigm| \phi(\cdot) \text{ is an } \mathrm{R}^{n\times d}\text{-valued } \mathscr{F}_t\text{-adapted cà dlàg process: } \mathbb{E}\left[\int_0^T \big|\phi(t)\big|^p\, dt\right] < +\infty\right\}.$$

If there is no risk of confusion, for notational simplicity, we write $\mathcal{L}^p_{\mathscr{F}_T} = \mathcal{L}^p_{\mathscr{F}_T}(\Omega; \mathrm{R}^k)$, $\mathcal{S}^p = \mathcal{S}^p(0, T; \mathrm{R}^n)$, $\mathcal{H}^p = \mathcal{H}^p(0, T; \mathrm{R}^{n\times d})$ and do not mention the concrete dimensions. For $x \in \mathrm{R}^n$, $y \in \mathrm{R}^{m\times d}$ and an R^k-valued vector function f on $\mathrm{R}^n \times \mathrm{R}^{m\times d}$, we use the notations

$$f_x \doteq \begin{pmatrix} \frac{\partial f^1}{\partial x^i} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^i} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^k}{\partial x^i} & \frac{\partial f^k}{\partial x^2} & \cdots & \frac{\partial f^k}{\partial x^n} \end{pmatrix}, \qquad f^i_y \doteq \begin{pmatrix} \frac{\partial f^i}{\partial y^{11}} & \frac{\partial f^i}{\partial y^{21}} & \cdots & \frac{\partial f^i}{\partial y^{m1}} \\ \frac{\partial f^i}{\partial y^{12}} & \frac{\partial f^i}{\partial y^{22}} & \cdots & \frac{\partial f^i}{\partial y^{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^i}{\partial y^{1d}} & \frac{\partial f^i}{\partial y^{2d}} & \cdots & \frac{\partial f^i}{\partial y^{md}} \end{pmatrix}.$$

Let U_i be a nonempty convex subset of R^{k_i} $(i = 1, 2)$. We define the admissible control set $\mathcal{U}_i$ by

$$\mathcal{U}_i = \left\{ v_i(\cdot) \;\middle|\; v_i(\cdot) : [0, T] \times \Omega \to U_i \text{ is an } \mathcal{F}_t\text{-adapted process} \right.$$
$$\left. \text{and satisfies that } \mathbb{E}\int_0^T v_i(t)^2\, dt < \infty \right\} \quad (i = 1, 2). \tag{9}$$

For each i, $\mathcal{U}_i$ is called an open-loop admissible control for Player i on $[0, T]$ $(i = 1, 2)$. $\mathcal{U}_1 \times \mathcal{U}_2$ is called the set of open-loop admissible controls for the players. For notational simplicity, hereinafter, we omit ω in u.

We introduce the mappings

$$\begin{aligned}
&f : [0, T] \times \mathrm{R}^n \times U_1 \times U_2 \to \mathrm{R}^n, \qquad \sigma : [0, T] \times \mathrm{R}^n \times U_1 \times U_2 \to \mathrm{R}^{n\times d}, \\
&g : [0, T] \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^m, \\
&\phi : \mathrm{R}^n \to \mathrm{R}^m, \qquad \varphi, \varphi_i : \mathrm{R}^n \to \mathrm{R}^1, \qquad \gamma, \gamma_i : \mathrm{R}^m \to \mathrm{R}^1, \\
&l, l_i : [0, T] \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^1 \quad (i = 1, 2).
\end{aligned}$$

Assumption (H1): f, σ, and ϕ are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and continuously differentiable with respect to (x, v_1, v_2). They are bounded by $(1 + |x| + |v_1| + |v_2|)$ and their derivatives with respect to (x, v_1, v_2) are continuous and uniformly bounded. g is uniformly Lipschitz in (y, z) and satisfies $\mathbb{E}\int_0^T |g(t, x, 0, 0, v_1(t), v_2(t))|^2\, dt < \infty$, $\forall x \in \mathrm{R}^n$, $v_i \in \mathcal{U}_i$.

Assumption (H2): $l, l_i, \varphi, \varphi_i, \gamma$ and γ_i $(i = 1, 2)$ are continuously differentiable with respect to (x, y, z, v_1, v_2). There exists a constant K_0 such that their partial derivatives with respect to (x, y, z, v_1, v_2) are bounded by $K_0(1 + |x| + |y| + |z| + |v_1| + |v_2|)$.

In the following, we specify the problem of nonzero-sum and zero-sum differential games of forward–backward stochastic systems, respectively. For simplicity, we denote them by *Problem I* and *Problem* LI, respectively.

Consider an FBSDE

$$\begin{cases} dx^{v_1,v_2}(t) = f\big(t, x^{v_1,v_2}(t), v_1(t), v_2(t)\big)\, dt + \sigma\big(t, x^{v_1,v_2}(t), v_1(t), v_2(t)\big)\, dB(t), \\ -dy^{v_1,v_2}(t) = g\big(t, x^{v_1,v_2}(t), y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\, dt - z^{v_1,v_2}(t)\, dB(t), \\ x^{v_1,v_2}(0) = x_0, \quad y^{v_1,v_2}(T) = \phi\big(x^{v_1,v_2}(T)\big), \quad 0 \leqslant t \leqslant T. \end{cases} \tag{10}$$

Consider a performance criterion

$$J_i\big(v_1(\cdot), v_2(\cdot)\big) = \mathbb{E}\left[\int_0^T l_i\big(t, x^{v_1,v_2}(t), y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\, dt + \varphi_i\big(x^{v_1,v_2}(T)\big)\right] + \gamma_i\big(y^{v_1,v_2}(0)\big) \tag{11}$$

with $l_i(\cdot, x^{v_1,v_2}(\cdot), y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{H}^1(0, T; \mathrm{R}^1)$ and $\varphi_i \in \mathcal{L}^1_{\mathcal{F}_T}(0, T; \mathrm{R})$ for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ $(i = 1, 2)$, and

$$J\big(v_1(\cdot), v_2(\cdot)\big) = \mathbb{E}\left[\int_0^T l\big(t, x^{v_1,v_2}(t), y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\, dt + \varphi\big(x^{v_1,v_2}(T)\big)\right] + \gamma\big(y^{v_1,v_2}(0)\big) \tag{12}$$

with $l(\cdot, x^{v_1,v_2}(\cdot), y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{H}^1(0, T; \mathrm{R}^1)$ and $\varphi \in \mathcal{L}^1_{\mathcal{F}_T}(0, T; \mathrm{R})$ for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$.

Under the assumptions (H1) and (H2), (10)–(12) are well posed. There are two players i_1 and i_2. Player i_1 controls v_1 and Player i_2 controls v_2. The nonzero-sum differential games are formulated as follows:

Problem I. Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1\big(u_1(\cdot), u_2(\cdot)\big) \geqslant J_1\big(v_1(\cdot), u_2(\cdot)\big), \\ J_2\big(u_1(\cdot), u_2(\cdot)\big) \geqslant J_2\big(u_1(\cdot), v_2(\cdot)\big), \end{cases} \tag{13}$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$.

We call $(u_1(\cdot), u_2(\cdot))$ an open-loop equilibrium point of *Problem I* (if it exists). It is easy to see that the existence of an open-loop equilibrium point implies

$$\begin{cases} J_1\big(u_1(\cdot), u_2(\cdot)\big) = \sup\limits_{v_1(\cdot)\in\mathcal{U}_1} J_1\big(v_1(\cdot), u_2(\cdot)\big), \\ J_2\big(u_1(\cdot), u_2(\cdot)\big) = \sup\limits_{v_2(\cdot)\in\mathcal{U}_2} J_2\big(u_1(\cdot), v_2(\cdot)\big). \end{cases}$$

The zero-sum differential games are formulated as follows:

Problem II. Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J\big(u_1(\cdot), v_2(\cdot)\big) \leqslant J\big(u_1(\cdot), u_2(\cdot)\big) \leqslant J\big(v_1(\cdot), u_2(\cdot)\big), \tag{14}$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$.

We call $(u_1(\cdot), u_2(\cdot))$ an open-loop saddle point of *Problem* II (if it exists). In fact the existence of an open-loop saddle point implies

$$\begin{aligned} J\big(u_1(\cdot), u_2(\cdot)\big) &= \sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) \\ &= \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big). \end{aligned}$$

We shall specify this point in Theorem 3.4(iii).

3. A maximum principle for differential games of FBSDEs

3.1. A maximum principle for nonzero-sum games

In this section, we establish a maximum principle for *Problem I*. Firstly, we apply the classical convex variational techniques to derive some variational inequalities. Then, we establish a necessary condition and a sufficient condition of an equilibrium point of *Problem I* respectively.

Suppose $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem I* with the trajectory $(x(\cdot), y(\cdot), z(\cdot))$ of (10). For all $t \in [0, T]$, let $v_i(t) \in U_i$ be such that $u_i(\cdot) + v_i(\cdot) \in \mathcal{U}_i$ $(i = 1, 2)$. Notice that $\mathcal{U}_i$ is convex, then for $0 \leqslant \rho \leqslant 1$, $i = 1, 2$,

$$u_{i\rho}(t) = u_i(t) + \rho v_i(t) \in \mathcal{U}_i, \quad 0 \leqslant t \leqslant T.$$

Assume that the process $(x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot))$ is the trajectory under $(u_{1\rho}(\cdot), u_{2\rho}(\cdot))$. For notational simplicity, we define the following symbol:

$$\begin{aligned} &f(t) \equiv f\big(t, x(t), u_1(t), u_2(t)\big), \qquad \sigma(t) \equiv \sigma\big(t, x(t), u_1(t), u_2(t)\big), \\ &g(t) \equiv g\big(t, x(t), y(t), z(t), u_1(t), u_2(t)\big), \qquad l(t) \equiv l\big(t, x(t), y(t), z(t), u_1(t), u_2(t)\big). \end{aligned}$$

We introduce the following variational equations:

$$\begin{cases} dx^i(t) = \big[f_x(t)x^i(t) + f_{v_i}(t)v_i(t)\big]\,dt + \big[\sigma_x(t)x^i(t) + \sigma_{v_i}(t)v_i(t)\big]\,dB(t), \\ -dy^i(t) = \big[g_x(t)x^i(t) + g_y(t)y^i(t) + g_z(t)z^i(t) + g_{v_i}(t)v_i(t)\big]\,dt - z^i(t)\,dB(t), \\ x^i(0) = 0, \quad y^i(T) = \phi_x\big(x(T)\big)x^i(T), \quad i = 1, 2. \end{cases} \tag{15}$$

For $i = 1, 2$, $t \in [0, T]$, $\rho > 0$, we set

$$\begin{aligned} &\tilde{x}^{i,\rho}(t) \triangleq \rho^{-1}\big(x_\rho(t) - x(t)\big) - x^i(t), \qquad \tilde{y}^{i,\rho}(t) \triangleq \rho^{-1}\big(y_\rho(t) - y(t)\big) - y^i(t), \\ &\tilde{z}^{i,\rho}(t) \triangleq \rho^{-1}\big(z_\rho(t) - z(t)\big) - z^i(t). \end{aligned}$$

By a linear method similar to Peng [14], Shi and Wu [20], we have the following convergence result:

Lemma 3.1. *Let assumption* (H1) *hold. Then it yields, for* $i = 1, 2$,

$$\lim_{\rho\to 0}\sup_{0\leqslant t\leqslant T}\mathbb{E}\left|\tilde{x}^{i,\rho}(t)\right|^2 = 0, \qquad \lim_{\rho\to 0}\sup_{0\leqslant t\leqslant T}\mathbb{E}\left|\tilde{y}^{i,\rho}(t)\right|^2 = 0,$$
$$\lim_{\rho\to 0}\mathbb{E}\int_0^T\left|\tilde{z}^{i,\rho}(t)\right|^2 dt = 0. \tag{16}$$

Proof. Firstly, we have

$$\begin{cases} d\tilde{x}^{i,\rho}(t) = \left[a_1^{i,\rho}(t)\tilde{x}^{i,\rho}(t) + a_2^{i,\rho}(t)\right]dt + \left[a_3^{i,\rho}(t)\tilde{x}^{i,\rho}(t) + a_4^{i,\rho}(t)\right]dB(t), \\ \tilde{x}^{i,\rho}(0) = 0, \end{cases}$$

where we denote $a_\cdot^{i,\rho}(t)$ as follows and omit the time subscript t in the right-hand of equality for simplicity

$$a_1^{1,\rho}(t) \triangleq \int_0^1 f_x\big(x + \lambda\rho\big(x^1 + \tilde{x}^{1,\rho}\big), u_1 + \lambda\rho v_1, u_2\big)\,d\lambda,$$

$$a_1^{2,\rho}(t) \triangleq \int_0^1 f_x\big(x + \lambda\rho\big(x^2 + \tilde{x}^{2,\rho}\big), u_1, u_2 + \lambda\rho v_2\big)\,d\lambda,$$

$$a_2^{1,\rho}(t) \triangleq \big[a_1^{1,\rho} - f_x(x, u_1, u_2)\big]x^1 + \int_0^1 f_{v_1}\big(x + \lambda\rho\big(x^1 + \tilde{x}^{1,\rho}\big), u_1 + \lambda\rho v_1, u_2\big)v_1\,d\lambda,$$

$$a_2^{2,\rho}(t) \triangleq \big[a_1^{2,\rho} - f_x(x, u_1, u_2)\big]x^2 + \int_0^1 f_{v_2}\big(x + \lambda\rho\big(x^2 + \tilde{x}^{2,\rho}\big), u_1, u_2 + \lambda\rho v_2\big)v_2\,d\lambda,$$

$$a_3^{1,\rho}(t) \triangleq \int_0^1 \sigma_x\big(x + \lambda\rho\big(x^1 + \tilde{x}^{1,\rho}\big), u_1 + \lambda\rho v_1, u_2\big)\,d\lambda,$$

$$a_3^{2,\rho}(t) \triangleq \int_0^1 \sigma_x\big(x + \lambda\rho\big(x^2 + \tilde{x}^{2,\rho}\big), u_1, u_2 + \lambda\rho v_2\big)\,d\lambda,$$

$$a_4^{1,\rho}(t) \triangleq \big[a_3^{1,\rho} - \sigma_x(x, u_1, u_2)\big]x^1 + \int_0^1 \sigma_{v_1}\big(x + \lambda\rho\big(x^1 + \tilde{x}^{1,\rho}\big), u_1 + \lambda\rho v_1, u_2\big)v_1\,d\lambda,$$

$$a_4^{2,\rho}(t) \triangleq \big[a_3^{2,\rho} - \sigma_x(x, u_1, u_2)\big]x^2 + \int_0^1 \sigma_{v_2}\big(x + \lambda\rho\big(x^2 + \tilde{x}^{2,\rho}\big), u_1, u_2 + \lambda\rho v_2\big)v_2\,d\lambda.$$

Applying Itô's formula to $|\tilde{x}^{i,\rho}(t)|^2$ and noting the assumption (H1), we have

$$\begin{aligned}\mathbb{E}\left|\tilde{x}^{i,\rho}(t)\right|^2 &= \mathbb{E}\int_0^T\left[\left\langle 2\tilde{x}^{i,\rho}(t), a_1^{i,\rho}(t)\tilde{x}^{i,\rho}(t) + a_2^{i,\rho}(t)\right\rangle + \left|a_3^{i,\rho}(t)\tilde{x}^{i,\rho}(t) + a_4^{i,\rho}(t)\right|^2\right]dt \\ &\leqslant C\mathbb{E}\int_0^T\left|\tilde{x}^{i,\rho}(t)\right|^2 dt + o(\rho).\end{aligned}$$

Then we can get the first convergence result of (16) from Gronwall's inequality.

Using the method as in the proof of the first convergent result in (16) and applying Itô's formula to $|\tilde{y}^{i,\rho}(t)|^2$, we have

$$\mathbb{E}\big|\tilde{y}^{i,\rho}(t)\big|^2+\mathbb{E}\int_t^T\big|\tilde{z}^{i,\rho}(s)\big|^2\,ds\leqslant C\mathbb{E}\int_t^T\big|\tilde{y}^{i,\rho}(s)\big|^2\,ds+\frac{1}{2}\mathbb{E}\int_t^T\big|\tilde{z}^{i,\rho}(s)\big|^2\,ds+o(\rho).$$

By Gronwall's inequality again, we get the last two convergent results of (16). □

Since $(u_1(\cdot),u_2(\cdot))$ is an equilibrium point, it follows that

$$\rho^{-1}\big[J_1\big(u_{1\rho}(\cdot),u_2(\cdot)\big)-J_1\big(u_1(\cdot),u_2(\cdot)\big)\big]\leqslant 0 \tag{17}$$

and

$$\rho^{-1}\big[J_2\big(u_1(\cdot),u_{2\rho}(\cdot)\big)-J_2\big(u_1(\cdot),u_2(\cdot)\big)\big]\leqslant 0 \tag{18}$$

are true. From (17), (18) and Lemma 3.1, we have

Lemma 3.2. *Let assumptions* (H1) *and* (H2) *hold. Then the following variational inequality holds for* $i=1,2$*:*

$$\begin{aligned}&\mathbb{E}\int_0^T\big[l_{ix}(t)x^i(t)+l_{iy}(t)y^i(t)+l_{iz}(t)z^i(t)+l_{iv_i}(t)v_i(t)\big]\,dt\\&\quad+\mathbb{E}\big[\varphi_{ix}\big(x(T)\big)x^i(T)\big]+\mathbb{E}\big[\gamma_{iy}\big(y(0)\big)y^i(0)\big]\leqslant 0.\end{aligned} \tag{19}$$

Proof. We firstly prove that (19) holds for $i=1$. From (H1), (H2) and Theorem 3.1, we derive the following

$$\begin{aligned}&\rho^{-1}\mathbb{E}\big[\varphi_1\big(x^{u_{1\rho},u_2}(T)\big)-\varphi_1\big(x(T)\big)\big]\\&\quad=\rho^{-1}\mathbb{E}\left[\int_0^1\varphi_{1x}\big(x(T)+\lambda\big(x^{u_{1\rho},u_2}(T)-x(T)\big)\big)\big(x^{u_{1\rho},u_2}(T)-x(T)\big)\,d\lambda\right]\\&\quad\to\mathbb{E}\big[\varphi_{1x}\big(x(T)\big)x^1(T)\big].\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}&\rho^{-1}\mathbb{E}\big[\gamma_1\big(y^{u_{1\rho},u_2}(0)\big)-\gamma_1\big(y(0)\big)\big]\\&\quad=\rho^{-1}\mathbb{E}\int_0^1\gamma_{1y}\big(y(0)+\lambda\big(y^{u_{1\rho},u_2}(0)-y(0)\big)\big)\big(y^{u_{1\rho},u_2}(0)-y(0)\big)\,d\lambda\\&\quad\to\mathbb{E}\big[\gamma_{1y}\big(y(0)\big)y^1(0)\big],\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\rho^{-1}\mathbb{E}\int_0^T\big[l_1\big(t,x^{u_{1\rho},u_2}(t),y^{u_{1\rho},u_2}(t),z^{u_{1\rho},u_2}(t),u_{1\rho}(t),u_2(t)\big)-l_1(t)\big]\,dt\\&\quad\to\mathbb{E}\int_0^T\big[l_{1x}(t)x^1(t)+l_{1y}(t)y^1(t)+l_{1z}(t)z^1(t)+l_{1v_1}(t)v^1(t)\big]\,dt.\end{aligned} \tag{22}$$

From (17) and (20)–(22), (19) follows for $i=1$. By the similar method above, (19) also holds for $i=2$. □

Next, before introducing the adjoint equations, we define the *Hamiltonian function* $H_i:[0,T]\times\mathrm{R}^n\times\mathrm{R}^m\times\mathrm{R}^{m\times d}\times\mathrm{U}_1\times\mathrm{U}_2\times\mathrm{R}^m\times\mathrm{R}^n\times\mathrm{R}^{n\times d}\to\mathrm{R}$ as follows:

$$\begin{aligned}&H_i(t,x,y,z,v_1,v_2,p_i,q_i,k_i)\\&\quad\triangleq\big\langle q_i,f(x,v_1,v_2)\big\rangle+\big\langle k_i,\sigma(x,v_1,v_2)\big\rangle-\big\langle p_i,g(x,y,z,v_1,v_2)\big\rangle+l_i(x,y,z,v_1,v_2).\end{aligned}$$

Let (v_1, v_2) and $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ be with the solutions $(x^{v_1,v_2}(\cdot), y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot))$ and $(x(\cdot), y(\cdot), z(\cdot))$ of Eq. (10), respectively. We denote

$$H_i^{v_1,v_2}(t) \equiv H_i\big(t, x^{v_1,v_2}(t), y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t), p_i^{v_1,v_2}(t), q_i^{v_1,v_2}(t), k_i^{v_1,v_2}(t)\big) \quad \text{and}$$
$$H_i(t) \equiv H_i\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_i(t), q_i(t), k_i(t)\big).$$

The adjoint equations satisfy the following generalized *stochastic Hamiltonian system*'s type:

$$\begin{cases} dp_i^{v_1,v_2}(t) = -H_{iy}^{v_1,v_2,*}(t)\,dt - H_{iz}^{v_1,v_2,*}(t)\,dB(t), \\ -dq_i^{v_1,v_2}(t) = H_{ix}^{v_1,v_2,*}(t)\,dt - k_i^{v_1,v_2}(t)\,dB(t), \\ p_i^{v_1,v_2}(0) = -\gamma_{iy}^*\big(y^{v_1,v_2}(0)\big), \\ q_i^{v_1,v_2}(T) = -\phi_{ix}^*\big(x^{v_1,v_2}(T)\big)p_i^{v_1,v_2}(T) + \varphi_{ix}^*\big(x^{v_1,v_2}(T)\big). \end{cases} \tag{23}$$

If the admissible controls are $(v_1(\cdot), v_2(\cdot))$ and $(u_1(\cdot), u_2(\cdot))$, we let $(p_i^{v_1,v_2}(\cdot), q_i^{v_1,v_2}(\cdot), k_i^{v_1,v_2}(\cdot))$ and $(p_i(\cdot), q_i(\cdot), k_i(\cdot))$ be the corresponding solutions of Eq. (23).

The main result of this paper is as follows:

Theorem 3.1 *(Necessary maximum principle). Let* (H1) *and* (H2) *hold. Let* $(u_1(\cdot), u_2(\cdot))$ *be an equilibrium point of Problem I with the corresponding solutions* $(x(\cdot), y(\cdot), z(\cdot))$ *and* $(p_i(\cdot), q_i(\cdot), k_i(\cdot))$ *of* (10) *and* (23)*. Then it follows that*

$$\big\langle H_{1v_1}^*(t), v_1 - u_1(t)\big\rangle \leqslant 0 \tag{24}$$

and

$$\big\langle H_{2v_2}^*(t), v_2 - u_2(t)\big\rangle \leqslant 0 \tag{25}$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$*, a.e., a.s.*

Proof. We firstly prove that (24) is true. Applying Itô's formula to $\langle x^1(t), q_1(t)\rangle + \langle y^1(t), p_1(t)\rangle$, we obtain the following result: $\forall v_1(\cdot) \in U_1$ such that $u_1(\cdot) + v_1(\cdot) \in \mathcal{U}_1$,

$$\begin{aligned} &\mathbb{E}\big[\varphi_{1x}\big(x(T)\big)x^1(T)\big] + \mathbb{E}\big[\gamma_{1y}\big(y(0)\big)y^1(0)\big] \\ &\quad = \mathbb{E}\int_0^T \big[-l_{1x}(t)x^1(t) - l_{1y}(t)y^1(t) - l_{1z}(t)z^1(t) - l_{1v_1}(t)v_1(t)\big]\,dt \\ &\qquad + \mathbb{E}\int_0^T \big\langle H_{1v_1}^*\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_1(t), q_1(t), k_1(t)\big)\big\rangle\,dt. \end{aligned}$$

This together with the variational inequality (19) derives that

$$\mathbb{E}\int_0^T \big\langle H_{1v_1}^*(t), v_1(t)\big\rangle\,dt \leqslant 0,$$

for all $t \in [0, T]$, $v_1(t) \in U_1$ s.t. $u_1(\cdot) + v_1(\cdot) \in \mathcal{U}_1$, which implies (24). By the similar method above, (25) also holds. □

Remark 3.1. It is very meaningful and important to seek an explicit equilibrium point for game players. In general, it is not easy to find this. This is because the solution variables of the forward–backward stochastic systems (10) and (23) are mutually coupled. In effect the state trajectory $(x^{v_1,v_2}, y^{v_1,v_2}, z^{v_1,v_2})$ in (10) depends on the control (v_1, v_2), and the adjoint process $(p_i^{v_1,v_2}, q_i^{v_1,v_2}, k_i^{v_1,v_2})$ in (23) depends on both the control (v_1, v_2) and the state $(x^{v_1,v_2}, y^{v_1,v_2}, z^{v_1,v_2})$. By the necessary maximum principle, Theorem 3.1, the control of the equilibrium point depends on both the state $(x^{v_1,v_2}, y^{v_1,v_2}, z^{v_1,v_2})$ and the adjoint process $(p_i^{v_1,v_2}, q_i^{v_1,v_2}, k_i^{v_1,v_2})$. So the foregoing procedures become a loop. In general, it is very hard to find an explicit equilibrium point. We may be able to find an explicit equilibrium point when the state equations and the cost functional are relatively simple.

In what follows, we proceed to establish the sufficient maximum principle (also called verification theorem). For this, we need an additional condition as follows:

(H3) $\phi(x) = Mx$ where M is a non-zero constant matrix with order $m \times n$. φ_i and γ_i are concave in x and y $(i = 1, 2)$, respectively.

Theorem 3.2 *(Sufficient conditions for optimality). Let* (H1), (H2) *and* (H3) *hold. Let* $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ *be with the corresponding solutions* (x, y, z) *and* (p_i, q_i, k_i) *of Eqs.* (10) *and* (23)*. Suppose*

$$\hat{H}_1(t, a, b, c) = \sup_{v_1 \in U_1} H_1\big(t, a, b, c, v_1, u_2(t), p_1(t), q_1(t), k_1(t)\big),$$
$$\hat{H}_2(t, a, b, c) = \sup_{v_2 \in U_2} H_2\big(t, a, b, c, u_1(t), v_2, p_1(t), q_1(t), k_1(t)\big) \tag{26}$$

exist for all $(t, a, b, c) \in [0, T] \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^{m \times d}$, *and are concave in* (a, b, c) *for all* $t \in [0, T]$ *(the Arrow condition). Moreover*

$$H_1\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_1(t), q_1(t), k_1(t)\big)$$
$$= \sup_{v_1 \in U_1} H_1\big(t, x(t), y(t), z(t), v_1, u_2(t), p_1(t), q_1(t), k_1(t)\big), \tag{27}$$
$$H_2\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_2(t), q_2(t), k_2(t)\big)$$
$$= \sup_{v_2 \in U_2} H_2\big(t, x(t), y(t), z(t), u_1(t), v_2, p_2(t), q_2(t), k_2(t)\big). \tag{28}$$

Then $(u_1(\cdot), u_2(\cdot))$ *is an equilibrium point of Problem I.*

Proof. Let $(v_1(\cdot), u_2(\cdot))$ and $(u_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ be with the corresponding solutions $(x^{v_1}, y^{v_1}, z^{v_1})$ and $(x^{v_2}, y^{v_2}, z^{v_2})$ to Eq. (10). We define the following terms

$$H_1(t) = H_1\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_1(t), q_1(t), k_1(t)\big),$$
$$H_1^{v_1}(t) = H_1\big(t, x^{v_1}, y^{v_1}, z^{v_1}, v_1(t), u_2(t), p_1(t), q_1(t), k_1(t)\big),$$
$$H_1^{v_2}(t) = H_1\big(t, x^{v_2}, y^{v_2}, z^{v_2}, u_1(t), v_2(t), p_1(t), q_1(t), k_1(t)\big),$$
$$f(t) = f\big(t, x(t), u_1(t), u_2(t)\big), \qquad f^{v_1}(t) = f\big(t, x^{v_1}, v_1(t), u_2(t)\big),$$
$$f^{v_2}(t) = f\big(t, x^{v_2}, u_1(t), v_2(t)\big),$$

and similarly define other terms $\sigma^{v_1}, \sigma^{v_2}, \ldots.$

By virtue of the concavity property of φ_1 and γ_1, we have for $\forall v_1(\cdot) \in \mathcal{U}_1$

$$J_1\big(v_1(\cdot), u_2(\cdot)\big) - J_1\big(u_1(\cdot), u_2(\cdot)\big) \leqslant I_1 + I_2 + I_3 \tag{29}$$

with

$$I_1 = \mathbb{E}\big[\gamma_{1y}\big(y(0)\big)\big(y^{v_1}(0) - y(0)\big)\big],$$
$$I_2 = \mathbb{E}\big[\varphi_{1x}\big(x(T)\big)\big(x^{v_1}(T) - x(T)\big)\big],$$
$$I_3 = \mathbb{E}\int_0^T \big(l^{v_1}(t) - l(t)\big)\,dt.$$

Applying Itô's formula to $\langle p_1(t), y^{v_1}(t) - y(t)\rangle$ and $\langle q_1(t), x^{v_1}(t) - x(t)\rangle$,

$$I_1 = -\mathbb{E}\big[\big\langle p_1(T), M\big(x^{v_1}(T) - x(T)\big)\big\rangle\big]$$
$$- \mathbb{E}\int_0^T \big(\big\langle p_1(t), g^{v_1}(t) - g(t)\big\rangle + \big\langle H_{1y}^*(t), y^{v_1}(t) - y(t)\big\rangle$$
$$+ \big\langle H_{1z}^*(t), z^{v_1}(t) - z(t)\big\rangle\big)\,dt, \tag{30}$$
$$I_2 = \mathbb{E}\int_0^T \big(\big\langle q_1(t), f^{v_1}(t) - f(t)\big\rangle + \big\langle k_1(t), \sigma^{v_1}(t) - \sigma(t)\big\rangle$$
$$- \big\langle H_{1x}^*(t), x^{v_1}(t) - x(t)\big\rangle\big)\,dt + \mathbb{E}\big[\big\langle p_1(T), M\big(x^{v_1}(T) - x(T)\big)\big\rangle\big], \tag{31}$$

$$I_3 = \mathbb{E}\int_0^T \big(H_1^{v_1}(t) - H_1(t) - \langle q_1(t), f^{v_1}(t) - f(t)\rangle \\ - \langle k_1(t), \sigma^{v_1}(t) - \sigma(t)\rangle + \langle p_1(t), g^{v_1}(t) - g(t)\rangle\big)\,dt. \tag{32}$$

Substituting (30)–(32) into (29), it follows immediately that

$$J_1\big(v_1(\cdot), u_2(\cdot)\big) - J_1\big(u_1(\cdot), u_2(\cdot)\big) \leqslant \mathbb{E}\int_0^T \big(H_1^{v_1}(t) - H_1(t) - \langle H_{1x}^*(t), x^{v_1}(t) - x(t)\rangle \\ - \langle H_{1y}^*(t), y^{v_1}(t) - y(t)\rangle - \langle H_{1z}^*(t), z^{v_1}(t) - z(t)\rangle\big)\,dt. \tag{33}$$

By virtue of (27) and the concavity of $\hat{H}_1$, we conclude that

$$J_1\big(v_1(\cdot), u_2(\cdot)\big) - J_1\big(u_1(\cdot), u_2(\cdot)\big) \leqslant 0, \tag{34}$$

for all $v_1(\cdot) \in \mathcal{U}_1$. By the similar method as shown in deriving (34), we can prove that

$$J_2\big(u_1(\cdot), v_2(\cdot)\big) - J_2\big(u_1(\cdot), u_2(\cdot)\big) \leqslant 0. \tag{35}$$

Based on the arguments above, $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem I*. □

Remark 3.2. For (26) to hold, it suffices that both functions

$$(a, b, c, v_1) \mapsto H_1\big(t, a, b, c, v_1, u_2(t), p_1(t), q_1(t), k_1(t)\big)$$

and

$$(a, b, c, v_2) \mapsto H_2\big(t, a, b, c, u_1, v_2(t), p_2(t), q_2(t), k_2(t)\big)$$

are concave for all $t \in [0, T]$.

3.2. A maximum principle for zero-sum games

In this section, we consider zero-sum differential games of FBSDEs. In fact, zero-sum games are a special case of nonzero-sum games. Based on the maximum principle of nonzero-sum games in Section 3.1, it is easy to obtain the maximum principle of zero-sum games.

Let

$$-l_1 = l_2 = l, \qquad -\varphi_1 = \varphi_2 = \varphi, \qquad -\gamma_1 = \gamma_2 = \gamma. \tag{36}$$

Then

$$H_1(t, x, y, z, v_1, v_2, p_1, q_1, k_1) \\ \triangleq \langle q_1, f(x, v_1, v_2)\rangle + \langle k_1, \sigma(x, v_1, v_2)\rangle - \langle p_1, g(x, y, z, v_1, v_2)\rangle - l(x, y, z, v_1, v_2), \tag{37}$$

$$H_2(t, x, y, z, v_1, v_2, p_2, q_2, k_2) \\ \triangleq \langle q_2, f(x, v_1, v_2)\rangle + \langle k_2, \sigma(x, v_1, v_2)\rangle - \langle p_2, g(x, y, z, v_1, v_2)\rangle + l(x, y, z, v_1, v_2), \tag{38}$$

and

$$-J_1 = J_2 = J.$$

We denote

$$H_1(t) \equiv H_1\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_1(t), q_1(t), k_1(t)\big)$$

and

$$H_2(t) \equiv H_2\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p_2(t), q_2(t), k_2(t)\big).$$

If $(u_1(\cdot), u_2(\cdot))$ is a saddle point of *Problem* II, we have

$$J\big(u_1(\cdot), v_2(\cdot)\big) \leqslant J\big(u_1(\cdot), u_2(\cdot)\big) \leqslant J\big(v_1(\cdot), u_2(\cdot)\big),$$

which implies that

$$\begin{cases} J_1\big(u_1(\cdot), u_2(\cdot)\big) \geqslant J_1\big(v_1(\cdot), u_2(\cdot)\big), \\ J_2\big(u_1(\cdot), u_2(\cdot)\big) \geqslant J_2\big(u_1(\cdot), v_2(\cdot)\big). \end{cases}$$

Therefore, $(u_1(\cdot), u_2(\cdot))$ is also an equilibrium point of *Problem I* with the special functions as in (36).

Applying the necessary condition of an equilibrium point in Theorem 3.1, we can directly derive the following necessary maximum principle of the foregoing zero-sum games.

Theorem 3.3 *(Necessary maximum principle). Let the assumptions* (H1) *and* (H2) *hold. Let* $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ *be a saddle point of Problem* ∐ *with corresponding solutions* (x, y, z) *and* (p, q, k) *of Eqs.* (10) *and* (23) *where the Hamiltonian functions* H_1 *and* H_2 *are defined by* (37) *and* (38) *respectively. Then it follows that*

$$\big\langle H^*_{1v_1}(t), v_1 - u_1(t)\big\rangle \leqslant 0, \tag{39}$$

and

$$\big\langle H^*_{2v_2}(t), v_2 - u_2(t)\big\rangle \leqslant 0 \tag{40}$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$*, a.e., a.s.*

Remark 3.3. Stochastic control problems can be regarded as zero-sum games with only one player. From this point, our results in this section can be considered partial extensions to the relevant results in Peng [14], Shi and Wu [19], Xu [24], Øksendal and Sulem [18], Shi and Wu [20]. For example, if the control $v_1(\cdot) \equiv 0$, the zero-sum game mentioned above is reduced to a stochastic optimal control problem of FBSDEs. The assumptions (H1) and (H2) together with the inequality (40) formulate a necessary maximum principle for optimal control of FBSDEs in a local form, which is the result of Peng [14].

Before we derive the sufficient condition of a saddle point, we first introduce some notations. The *Hamiltonian function* $H : [0, T] \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times \mathrm{U}_1 \times \mathrm{U}_2 \times \mathrm{R}^m \times \mathrm{R}^n \times \mathrm{R}^{n\times d} \to \mathrm{R}$ is defined as follows:

$$\begin{aligned} &H(t, x, y, z, v_1, v_2, p, q, k) \\ &\quad \triangleq \big\langle q, f(x, v_1, v_2)\big\rangle + \big\langle k, \sigma(x, v_1, v_2)\big\rangle - \big\langle p, g(x, y, z, v_1, v_2)\big\rangle + l(x, y, z, v_1, v_2). \end{aligned}$$

We denote

$$H^{v_1,v_2}(t) \equiv H\big(t, x^{v_1,v_2}(t), y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t), p^{v_1,v_2}(t), q^{v_1,v_2}(t), k^{v_1,v_2}(t)\big)$$

and

$$H(t) \equiv H\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p(t), q(t), k(t)\big).$$

The adjoint equations satisfy the following *stochastic Hamiltonian system*'s type:

$$\begin{cases} dp^{v_1,v_2}(t) = -H_y^{v_1,v_2,*}(t)\,dt - H_z^{v_1,v_2,*}(t)\,dB(t), \\ -dq^{v_1,v_2}(t) = H_x^{v_1,v_2,*}(t)\,dt - k^{v_1,v_2}(t)\,dB(t), \\ p^{v_1,v_2}(0) = -\gamma_y^*\big(y^{v_1,v_2}(0)\big), \\ q^{v_1,v_2}(T) = -\phi_x^*\big(x^{v_1,v_2}(T)\big)p^{v_1,v_2}(T) + \varphi_x^*\big(x^{v_1,v_2}(T)\big). \end{cases} \tag{41}$$

Then we have the following sufficient maximum principle of zero-sum games.

Theorem 3.4 *(Sufficient conditions for optimality). Let* (H_1), (H_2) *and* (H_3) *hold. Let* $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ *be with the corresponding solutions* (x, y, z) *and* (p, q, k) *of Eqs.* (10) *and* (41)*. Suppose that the Hamiltonian function H satisfies the following conditional mini-maximum principle*:

$$\begin{aligned} &\inf_{v_1(\cdot)\in\mathcal{U}_1} H\big(t, x(t), y(t), z(t), v_1(t), u_2(t), p(t), q(t), k(t)\big) \\ &\quad = H\big(t, x(t), y(t), z(t), u_1(t), u_2(t), p(t), q(t), k(t)\big) \\ &\quad = \sup_{v_2(\cdot)\in\mathcal{U}_2} H\big(t, x(t), y(t), z(t), u_1(t), v_2(t), p(t), q(t), k(t)\big). \end{aligned} \tag{42}$$

(i) *Assume that both φ and γ are concave, and*

$$\hat{H}_2(t,a,b,c) = \sup_{v_2(\cdot)\in\mathcal{U}_2} H\big(t,a,b,c,u_1(t),v_2,p(t),q(t),k(t)\big)$$

exists for all $(t,a,b,c)\in[0,T]\times \mathrm{R}^n\times \mathrm{R}^m\times \mathrm{R}^{m\times d}$, and is concave in (a,b,c). Then we have

$$J\big(u_1(\cdot),v_2(\cdot)\big) \leqslant J\big(u_1(\cdot),u_2(\cdot)\big), \quad \textit{for all } v_2(\cdot)\in\mathcal{U}_2,$$

and

$$J\big(u_1(\cdot),u_2(\cdot)\big) = \sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(u_1(\cdot),v_2(\cdot)\big).$$

(ii) *Assume that both φ and γ are convex, and*

$$\hat{H}_1(t,a,b,c) = \inf_{v_1(\cdot)\in\mathcal{U}_1} H\big(t,a,b,c,v_1,u_2(t),p(t),q(t),k(t)\big)$$

exists for all $(t,a,b,c)\in[0,T]\times \mathrm{R}^n\times \mathrm{R}^m\times \mathrm{R}^{m\times d}$, and is convex in (a,b,c). Then we have

$$J\big(u_1(\cdot),u_2(\cdot)\big) \leqslant J\big(v_1(\cdot),u_2(\cdot)\big), \quad \textit{for all } v_1(\cdot)\in\mathcal{U}_1,$$

and

$$J\big(u_1(\cdot),u_2(\cdot)\big) = \inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),u_2(\cdot)\big).$$

(iii) *If both* (i) *and* (ii) *are true, then $(u_1(\cdot),u_2(\cdot))$ is a saddle point which implies*

$$\begin{aligned}\sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big) &= J\big(u_1(\cdot),u_2(\cdot)\big)\\ &= \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot),v_2(\cdot)\big)\Big).\end{aligned}$$

Proof. (i) Using the same argument as in the proof of Theorem 3.2, we can obtain the following:

$$J\big(u_1(\cdot),v_2(\cdot)\big) \leqslant J\big(u_1(\cdot),u_2(\cdot)\big), \quad \text{for all } v_2(\cdot)\in\mathcal{U}_2. \tag{43}$$

Furthermore

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(u_1(\cdot),v_2(\cdot)\big) \leqslant J\big(u_1(\cdot),u_2(\cdot)\big).$$

Since $u_2(\cdot)\in\mathcal{U}_2$, we have

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(u_1(\cdot),v_2(\cdot)\big) = J\big(u_1(\cdot),u_2(\cdot)\big).$$

(ii) This statement can be proved in a similar way as shown before.
(iii) If both (i) and (ii) are true, then

$$J\big(u_1(\cdot),v_2(\cdot)\big) \leqslant J\big(u_1(\cdot),u_2(\cdot)\big) \leqslant J\big(v_1(\cdot),u_2(\cdot)\big),$$

for all $(v_1(\cdot),v_2(\cdot))\in\mathcal{U}_1\times\mathcal{U}_2$, i.e., $(u_1(\cdot),u_2(\cdot))$ is a saddle point.
In the following, on the one hand, we have

$$J\big(u_1(\cdot),u_2(\cdot)\big) \leqslant \inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),u_2(\cdot)\big) \leqslant \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot)v_2(\cdot)\big)\Big),$$

and

$$J\big(u_1(\cdot),u_2(\cdot)\big) \geqslant \sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(u_1(\cdot),v_2(\cdot)\big) \geqslant \sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big),$$

which imply that

$$\begin{aligned}\sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big) &\leqslant J\big(u_1(\cdot),u_2(\cdot)\big)\\ &\leqslant \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot),v_2(\cdot)\big)\Big).\end{aligned} \tag{44}$$

On the other hand, we have

$$J\big(u_1(\cdot),u_2(\cdot)\big)\leqslant \inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),u_2(\cdot)\big)\leqslant \sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big)$$

and

$$J\big(u_1(\cdot),u_2(\cdot)\big)\geqslant \sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(u_1(\cdot),v_2(\cdot)\big)\geqslant \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot),v_2(\cdot)\big)\Big),$$

which imply that

$$\sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big)\geqslant J\big(u_1(\cdot),u_2(\cdot)\big)\geqslant \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot),v_2(\cdot)\big)\Big). \tag{45}$$

Combining (44) and (45), we have

$$\sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot),v_2(\cdot)\big)\Big)= J\big(u_1(\cdot),u_2(\cdot)\big)= \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot),v_2(\cdot)\big)\Big). \qquad \square$$

4. An example of a nonzero-sum game of FBSDE

In this section, we work out an example of nonzero-sum differential games of FBSDEs to illustrate our theoretical result. Firstly, by applying the necessary optimality condition (see Theorem 3.1), we find an explicit candidate equilibrium point. Then by the sufficient condition of an equilibrium point (see Theorem 3.2), we verify that it is indeed an equilibrium point. Finally, we obtain the explicit state trajectories and the explicit value of the performance criteria under the control of the equilibrium point.

Example 4.1. Consider the system of FBSDE

$$\begin{cases} dx^{v_1,v_2}(t)=\big[a_1(t)x^{v_1,v_2}(t)+a_2(t)v_1(t)+\bar{a}_2(t)v_2(t)\big]dt+\big[a_3(t)x^{v_1,v_2}(t)\big]dB(t),\\ -dy^{v_1,v_2}(t)=\big[b_1(t)x^{v_1,v_2}(t)+b_2(t)v_1(t)+\bar{b}_2(t)v_2(t)\\ \qquad\qquad +b_3(t)y^{v_1,v_2}(t)+b_4(t)z^{v_1,v_2}(t)\big]dt-z^{v_1,v_2}(t)\,dB(t),\\ x^{v_1,v_2}(0)=x_0,\quad y^{v_1,v_2}(T)=x^{v_1,v_2}(T), \end{cases} \tag{46}$$

with the performance criteria

$$J_1\big(v_1(\cdot),v_2(\cdot)\big)=\mathbb{E}\Bigg[\int_0^T\Big(-\frac{1}{2}v_1^2(t)+d_1(t)x^{v_1,v_2}(t)\Big)dt+e_1y^{v_1,v_2}(0)\Bigg] \tag{47}$$

and

$$J_2\big(v_1(\cdot),v_2(\cdot)\big)=\mathbb{E}\Bigg[\int_0^T\Big(-\frac{1}{2}v_2^2(t)+d_2(t)y^{v_1,v_2}(t)\Big)dt+e_2x^{v_1,v_2}(T)\Bigg]. \tag{48}$$

Here, a_i, $\bar{a}_i$, b_i, $\bar{b}_i$, d_i and e_i are bounded and deterministic. The set of admissible controls is defined by

$$\mathcal{U}_i=\Bigg\{v_i(\cdot)\;\Bigg|\;v_i(\cdot)\text{ is an }\mathrm{R}^1\text{-valued }\mathscr{F}_t\text{-adapted process and satisfies }\mathbb{E}\int_0^T v_i^2(t)\,dt<\infty\Bigg\},\quad i=1,2. \tag{49}$$

Then $(u_1(\cdot),u_2(\cdot))$ in (56) is an equilibrium point, such that

$$\begin{cases} J_1\big(u_1(\cdot),u_2(\cdot)\big)=\sup\limits_{v_1(\cdot)\in\mathcal{U}_1} J_1\big(v_1(\cdot),u_2(\cdot)\big),\\ J_2\big(u_1(\cdot),u_2(\cdot)\big)=\sup\limits_{v_2(\cdot)\in\mathcal{U}_2} J_2\big(u_1(\cdot),v_2(\cdot)\big). \end{cases} \tag{50}$$

Proof. (i) $(u_1(\cdot),u_2(\cdot))$ in (56) is a candidate equilibrium point.

In order to apply the necessary maximum principle, we first write down the *Hamiltonian* function

$$\begin{aligned} H_1(t,x,y,z,v_1,v_2,p_1,q_1,k_1)\triangleq{}& q_1\big(a_1(t)x+a_2(t)v_1+\bar{a}_2(t)v_2\big)+k_1a_3(t)x-\frac{1}{2}v_1^2+d_1(t)x\\ &-p_1\big(b_1(t)x+b_2(t)v_1+\bar{b}_2(t)v_2+b_3(t)y+b_4(t)z\big) \end{aligned} \tag{51}$$

where $(p_1(\cdot), q_1(\cdot), k_1(\cdot))$ satisfies the following adjoint equation

$$\begin{cases} dp_1(t) = b_3(t)p_1(t)\,dt + b_4(t)p_1(t)\,dB(t), \\ -dq_1(t) = \big[a_1(t)q_1(t) + a_3(t)k_1(t) - b_1(t)p_1(t) + d_1(t)\big]\,dt - k_1(t)\,dB(t), \\ p_1(0) = e_1, \quad q_1(T) = -p_1(T). \end{cases} \tag{52}$$

Similarly, we have

$$\begin{aligned} H_2(t, x, y, z, v_1, v_2, p_2, q_2, k_2) \triangleq{}& q_2\big(a_1(t)x + a_2(t)v_1 + \bar{a}_2(t)v_2\big) + k_2 a_3(t)x - \frac{1}{2}v_2^2 + d_2(t)y \\ & - p_2\big(b_1(t)x + b_2(t)v_1 + \bar{b}_2(t)v_2 + b_3(t)y + b_4(t)z\big) \end{aligned} \tag{53}$$

where $(p_2(\cdot), q_2(\cdot), k_2(\cdot))$ satisfies the following adjoint equation

$$\begin{cases} dp_2(t) = \big(b_3(t)p_2(t) - d_2(t)\big)\,dt + b_4(t)p_2(t)\,dB(t), \\ -dq_2(t) = \big[a_1(t)q_2(t) + a_3(t)k_2(t) - b_1(t)p_2(t)\big]\,dt - k_2(t)\,dB(t), \\ p_2(0) = 0, \quad q_1(T) = -p_2(T) + e_2. \end{cases} \tag{54}$$

By solving the decoupled FBSDEs (52) and (54), we obtain the following explicit solutions

$$\begin{cases} p_1(t) = e_1 \exp\left\{ \int_0^t \left(b_3(s) - \frac{1}{2}b_4(s)^2 \right) ds + \int_0^t b_4(s)\,dB(s) \right\}, \\ q_1(t) = \mathbb{E}\left[-p_1(T)\Gamma_1^t(T) + \int_t^T \Gamma_1^t(s)\big(-b_1(s)p_1(s) + d_1(s)\big)\,ds \,\middle|\, \mathscr{F}_t \right], \\ p_2(t) = -\int_0^t d_2(s) \exp\left\{ \int_s^t \left(b_3(r) - \frac{1}{2}b_4(r)^2 \right) dr + \int_s^t b_4(r)\,dB(r) \right\} ds, \\ q_2(t) = \mathbb{E}\left[\big(-p_2(T) + e_2\big)\Gamma_1^t(T) - \int_t^T \Gamma_1^t(s) b_1(s) p_2(s)\,ds \,\middle|\, \mathscr{F}_t \right], \end{cases} \tag{55}$$

where

$$\Gamma_1^t(s) = 1 + \int_t^s \Gamma_1^t(r) a_1(r)\,dr + \int_t^s \Gamma_1^t(r) a_3(r)\,dB(r), \quad s \geqslant t.$$

By virtue of the necessary maximum principle (Theorem 3.1), we obtain a candidate equilibrium point $(u_1(\cdot), u_2(\cdot))$ with

$$\begin{cases} u_1(t) = a_2(t)q_1(t) - b_2(t)p_1(t) \\ \qquad = a_2(t)\mathbb{E}\left[-p_1(T)\Gamma_1^t(T) + \int_t^T \Gamma_1^t(s)\big(-b_1(s)p_1(s) + d_1(s)\big)\,ds \,\middle|\, \mathscr{F}_t \right] \\ \qquad\quad - b_2(t)e_1 \exp\left\{ \int_0^t \left(b_3(s) - \frac{1}{2}b_4(s)^2 \right) ds + \int_0^t b_4(s)\,dB(s) \right\}, \\ u_2(t) = \bar{a}_2(t)q_2(t) - \bar{b}_2(t)p_2(t) \\ \qquad = \bar{a}_2(t)\mathbb{E}\left[\big(-p_2(T) + e_2\big)\Gamma_1^t(T) - \int_t^T \Gamma_1^t(s) b_1(s) p_2(s)\,ds \,\middle|\, \mathscr{F}_t \right] \\ \qquad\quad + \bar{b}_2(t) \int_0^t d_2(s) \exp\left\{ \int_s^t \left(b_3(r) - \frac{1}{2}b_4(r)^2 \right) dr + \int_s^t b_4(r)\,dB(r) \right\} ds, \end{cases} \tag{56}$$

where $(p_1(\cdot), p_2(\cdot))$ is denoted by (55).

(ii) $(u_1(\cdot), u_2(\cdot))$ in (56) is indeed an equilibrium point.

We can check that $\gamma_1(y) = e_1 y$, $\gamma_2(y) = 0$, $\varphi_1(x) = 0$, $\varphi_2(x) = e_2 x$, $\phi(x) = x$ are concave. $H_1(t, x, y, z, v_1, v_2, p_1, q_1, k_1)$ and $H_2(t, x, y, z, v_1, v_2, p_2, q_2, k_2)$ in (51) and (53) are concave in (x, y, z, v_1, v_2), respectively. Meanwhile H_1 and H_2 satisfy the conditions (27) and (28), respectively. Applying the sufficient maximum principle (Theorem 3.2), $(u_1(\cdot), u_2(\cdot))$ proves an equilibrium point indeed. $\square$

In the following, we obtain the explicit state trajectories $(x(\cdot), y(\cdot))$ under the control $(u_1(\cdot), u_2(\cdot))$ in (56). For notational simplicity, we define

$$\Gamma_2(t) \triangleq \exp\left\{ \int_0^t \left(a_1(s) - \frac{1}{2} a_3(s)^2 \right) ds + \int_0^t a_3(s)\, dB(s) \right\},$$

$$\Gamma_3^t(s) \triangleq 1 + \int_t^s \Gamma_3^t(r) b_3(r)\, dr + \int_t^s \Gamma_3^t(r) b_4(r)\, dB(r), \quad s \geqslant t.$$

By the classical multiplier factor method, we solve the system (46) and obtain the following explicit unique solution:

$$\begin{cases} x(t) = \Gamma_2(t)\left\{ x_0 + \displaystyle\int_0^t \Gamma_2^{-1}(s)\big(a_2(s)u_1(s) + \bar{a}_2(s)u_2(s)\big)\, ds \right\}, \\ y(t) = \mathbb{E}\left\{ x(T)\Gamma_3^t(T) + \displaystyle\int_t^T \Gamma_3^t(r)\big[b_2(r)u_1(r) + \bar{b}_2(r)u_2(r) + b_1(r)x(r)\big]\, dr \,\middle|\, \mathscr{F}_t \right\} \\ \quad = \mathbb{E}\left\{ \Gamma_2(T)\left(x_0 + \displaystyle\int_0^T \Gamma_2^{-1}(s)\big(a_2(s)u_1(s) + \bar{a}_2(s)u_2(s)\big)\, ds \right)\Gamma_3^t(T) \right. \\ \qquad + \displaystyle\int_t^T \Gamma_3^t(r)\left[b_2(r)u_1(r) + \bar{b}_2(r)u_2(r) \right. \\ \qquad \left.\left. + b_1(r)\Gamma_2(r)\left(x_0 + \displaystyle\int_0^r \Gamma_2^{-1}(s)\big(a_2(s)u_1(s) + \bar{a}_2(s)u_2(s)\big)\, ds \right) \right] dr \,\middle|\, \mathscr{F}_t \right\}, \end{cases} \tag{57}$$

where $(u_1(\cdot), u_2(\cdot))$ is defined by (56). Substituting (57) into the performance criteria (47) and (48), we can obtain the values of $J_1(u_1(\cdot), u_2(\cdot))$ and $J_2(u_1(\cdot), u_2(\cdot))$.

5. Conclusion

Motivated by LQ differential game with generalized expectation, this paper has investigated differential games of forward–backward stochastic systems. By applying the classical convex variational techniques and the FBSDEs theory, we established a necessary condition and a sufficient condition for equilibrium points of nonzero-sum games. Since zero-sum games can be regarded as a special case of nonzero-sum games, we also derived the corresponding conditions for the saddle point of zero-sum games. Compared with Buckdahn and Li [3], the performance criterion we considered is more general, with a processing method different from their dynamic programming. Our result is an extension to Wang and Yu [22] and Yu and Ji [30] for backward systems and Yu [29] for linear systems. Stochastic control problems can be considered as a special case of zero-sum games with only one player. From this point of view, our results about zero-sum games represent partial extensions to the relevant results in Peng [14], Shi and Wu [19], Xu [24], Øksendal and Sulem [18], Shi and Wu [20] for stochastic control problems with forward–backward systems. To illustrate possible applications to theoretical results, this paper has further worked out an example of a nonzero-sum differential game. We used the necessary condition to find an explicit candidate equilibrium point, and then employed the sufficient one to verify that it is indeed a unique equilibrium point. Moreover, the explicit state trajectories and performance criteria have been obtained.

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